

GLOBAL FUNCTIONS IN GLOBAL-LOCAL FINITE-ELEMENT ANALYSIS
OF
LOCALIZED STRESSES IN PRISMATIC STRUCTURES

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Abstract

An important consideration in the global local finite-element method (GLFEM) is the availability of global functions for the given problem. The role and mathematical requirements of these global functions in a GLFEM analysis of localized stress states in prismatic structures are discussed. A method is described for determining these global functions. Underlying this method are theorems due to Toupin and Knowles on strain energy decay rates, which are related to a quantitative expression of Saint-Venant's principle. It is mentioned that a mathematically complete set of global functions can be generated, so that any arbitrary interface condition between the finite element and global subregions can be represented. Convergence to the true behavior can be achieved with increasing global functions and finite-element degrees of freedom. Specific attention is devoted to mathematically two-dimensional and three-dimensional prismatic structures. Comments are offered on the GLFEM analysis of NASA flat panel with a discontinuous stiffener. Methods for determining global functions for other effects are also indicated, such as steady-state dynamics and bodies under initial stress.

Introduction

The finite-element method (FEM) has revolutionized structural and stress analyses in the last quarter of this century. Its impact has been widespread, even extending beyond the preserve of structural engineers to other fields. Although FEM is acknowledged as an extremely powerful modeling technique, the analysis community with its collective experience will admit that it is not the quintessential technique. There are problems not well suited to FEM that result in clumsy, ineffective and costly mathematical models. Examples can be cited from problems involving stress singularities and infinite domains. To obviate the difficulties, modifications to FEM have been explored. One alternate approach which bodes considerable promise is the so-called Global-Local Finite-Element Method (GLFEM).

GLFEM utilizes both conventional finite elements and classical Ritz functions in the modeling process. Their respective roles are readily apparent; finite elements work well in regions where complicated geometry and inhomogeneous material characterizations prevail, and Ritz functions, hereinafter referred to as global functions in GLFEM, enable the behavior of other regions to be represented accurately and efficiently. At this stage of development, GLFEM can be assessed to be in its maturing phase. It is of good lineage, has already exhibited an enhanced capability above FEM in certain problems, and promises effectiveness in other classes of problems upon its full development.

Herein, GLFEM as applied to the analysis of localized stresses in prismatic structures is discussed. First, the essence of GLFEM and various GLFEM modeling layouts are summarized. A brief review of some problems that have been successfully analyzed by GLFEM is given. Then, the main theme relating to GLFEM analysis of localized stress states is addressed. Prismatic structures that can be described mathematically by two spatial variables are discussed first. Attention is devoted to the global functions, their development and their roles in the present setting. Then, three-dimensional structures are considered, with reference to the NASA example problem, where an outline of a method of attack is given. Last, comments on the analysis of localized stresses involving steady-state dynamic effects as well as other conditions are given.

Basic Concepts of GLFEM and the Various Mesh Configurations

Hamilton's principle, or alternatively the theorem of minimum potential energy when no inertial effects are present, may be considered as the basis for generating GLFEM equations. The theory and variational derivation of these equations may be found in Ref. [1, pp.451-474]. Also included therein is a survey of GLFEM contributions to the literature up to 1982.

As noted earlier, the technique utilizes finite-element modeling with classical Ritz approximations simultaneously. It enjoys the advantages of

more versatile modeling capabilities with substantially fewer degrees of freedom. Various global/local modeling configurations are illustrated in Fig. 1. Figs. 1a and 1f represent, respectively, the classical Ritz and finite-element configurations. The others are possible GLFEM mesh layouts. In a given problem, the modeling may take the form of any one of these configurations or a combination of two or more of them for various subregions. An important key is the enforcement of kinematic inter-regional continuity between various global and local subregions by means of constraint equations. In problems on localized stress states, only the Fig. 1c configuration will be used, where finite elements exclusively are used in one sub-region and global functions in the other. Moreover, the global subregion may be infinite in extent.

The governing matrix equations in a GLFEM analysis have the form:

$$\begin{bmatrix} [K_{ll}] & [K_{lg}] \\ [K_{gl}] & [K_{gg}] \end{bmatrix} \begin{bmatrix} \{\delta\} \\ \{S\} \end{bmatrix} + \begin{bmatrix} [M_{ll}] & [M_{lg}] \\ [M_{gl}] & [M_{gg}] \end{bmatrix} \begin{bmatrix} \{\delta\} \\ \{S\} \end{bmatrix} = \begin{bmatrix} \{F_l\} \\ \{F_g\} \end{bmatrix} \quad (1)$$

where $\{\delta\}$ denotes the finite-element degrees of freedom and $\{S\}$ contains the array of generalized coordinates associated with the global functions. In Eq. (1), $[K_{gg}]$, $[M_{gg}]$, $[K_{ll}]$, and $[M_{ll}]$ refer to the global and local stiffness and mass matrices of the system. The matrices $[K_{lg}] = [K_{lg}]^T$ and $[M_{lg}] = [M_{lg}]^T$ represent global-local coupling from imposing kinematic continuity at interface(s) between subregion(s). Details on the formation of these matrices may be found in Ref. [1].

It is mentioned that GLFEM variants are possible, which do not lead to the same set of governing equations as Eq. (1). These variants contain the spirit of GLFEM and employ the modeling configurations shown in Fig. 1; however, the method of enforcing inter-regional continuity may differ. An application concerned with elastic wave scattering will illustrate one such variant.

Another key point in GLFEM is the availability of an appropriate set of global functions for a given problem or a class of problems. The accuracy and effectiveness of the method are dependent upon the quality of the global functions. The choice of these global functions for the analysis of localized stresses in prismatic structures and their method of derivation will be discussed in what follows. It will become apparent why these global functions, together with the finite-element model of the subregion that contains the localized stresses, will lead to a superior model.

Some Examples of Global Functions for GLFEM

Two areas ideally suited to GLFEM are fracture mechanics and infinite

and/or semi-infinite domain problems. Much has been published on various aspects of fracture mechanics problems. Many numerical methods have been used, many falling within a GLFEM classification or its variants. The global subregion model usually takes the form of special crack tip elements, where the singular stress field is incorporated into the stiffness matrix. These elements are well known. No further elaboration on this subject will be given here. Elastostatic analyses of half-space problems, wherein the far field behavior is represented by global functions (for example, Boussinesq or Cerruti solutions), have also met with considerable success. A number of references on both of these subjects may be found in Ref. [1].

Herein, two recent GLFEM applications are mentioned to emphasize the roles of the global functions and their mathematical suitability. They are concerned with (1) steady-state elastic wave scattering by axisymmetric objects embedded in an infinite isotropic medium and (2) steady-state soil-structure interaction involving an axisymmetric structure occupying some locale in a semi-infinite medium. The feature of note is that these global functions constitute a complete set of eigenfunctions and have the capability of mathematically representing an arbitrary scattered field to any given accuracy. Hence, the true behavior in the far field can be achieved.

In Fig. 2 is shown an elastic, axisymmetric inclusion embedded in an elastic, isotropic medium. Because finite elements are used for the object it may have inhomogeneous, orthotropic properties. The finite-element subregion includes this object and a portion of the surrounding medium. For convenience in the analysis, the interface is taken to be spherical. Outside of the finite-element subregion is the outer field, where a complete set of outgoing spherical harmonics is used to model the scattered field. Each component satisfies the equations of motion and the Sommerfeld radiation conditions. The global functions have specific stress and displacement distributions at the interface, and their undetermined strengths are the global function coefficients or the terms in $\{S\}$. A given incident wave illuminates this object. The scattered field is determined by solving the finite-element equations and requiring that the sum of incident and scattered wave fields based on the global functions have both traction and displacement continuity with the finite-element data at the interface. Details of this analysis may be found in Ref. [2]. Here, attention is called to the mathematical flexibility of the global functions for accommodating interface continuity to any precision with a sufficient number of terms.

The dynamic soil-structure interaction problem under steady-state conditions is shown in Fig. 3. The approach used here is similar to that for elastic wave scattering by an object embedded in the entire space. In fact, the same set of spherical harmonics for the entire space may be applied to this half-space problem. However, traction-free surface conditions are not satisfied by the spherical harmonics. Thus, in addition to traction and displacement continuity at the hemispherical interface, it is necessary to enforce the traction-free surface in the global subregion. In Refs. [3,4], details concerning an integral

constraint condition to meet this traction-free surface condition are given. Again, it is noted that because a complete set of global functions is used (that is, a set capable of modeling any arbitrary traction and displacement conditions between various subregions in a GLFEM layout), the analysis procedure enjoys the opportunity of converging onto the true behavior with increasing FEM and global degrees of freedom.

Mathematically Two-Dimensional Structures

The choice of global functions for mathematically two-dimensional structures will now be discussed. As illustrations of this class of problems and their GLFEM layouts, refer to Fig. 4, where examples of a laminated composite plate and cylinders are given. The double lap joint may be considered as a plane strain problem herein. The scarf joint joining two cylinders may be taken as an axisymmetric structure under axisymmetric or asymmetric loads. The purpose is to study the stresses in these joints.

Uniform stress states exist at points well away from these localized stress regions. If FEM were used, it is obvious that an awkward model would result. In GLFEM, two-dimensional finite elements (planar or axisymmetric toroidal elements) are used for the subregion containing the localized stresses. If the localized stress state contains a singularity, a global subregion within the finite-element subregion may be added. The interface location is dependent on the global functions' mathematical capability for capturing the transitional stress and displacement fields accurately. For global functions capable of representing the true behavior, the finite-element subregion can be quite small with the interface(s) near to the localized stress area. An independent set of global functions must be adopted at each interface. For the lap joint in Fig. 4, two or three distinct systems of global functions may be needed depending on the thickness and material properties of the plate components. Each set of global functions is associated with its own set of generalized coordinates or global coefficients. For the cylindrical scarf joint, two independent sets are needed.

The global functions in these cases are based on theorems relating to a quantitative expression of St. Venant's principle. Toupin [5] and Knowles [6] presented upper bound estimates of strain energy decay rates in terms of distance from a self-equilibrated stress state. Their results can be stated in the form of a strain energy inequality:

$$V(x) \leq V(0) e^{-2\gamma x} \quad (2)$$

where γ is the inverse of the characteristic decay length, $V(0)$ is the total strain energy and $V(x)$ is that portion of $V(0)$ in the body beyond x . Since the strain energy is quadratic, the mechanical variables such as stress, strain and displacement are of the forms:

$$u_j(x) \leq K_1 e^{-\gamma x} \quad \epsilon_{ij}(x) \leq K_2 e^{-\gamma x} \quad \sigma_{ij}(x) \leq K_3 e^{-\gamma x} \quad (3)$$

where K_i ($i=1,2,3$) are constants.

Based on these theorems, a boundary-value problem can be formulated for a prismatic structure. Using the solution form in the prismatic direction as $e^{-\gamma x}$, the analysis leads to an eigenvalue problem. The eigenvalues γ 's are the characteristic inverse decay lengths and the eigenfunctions are distributions of self-equilibrated stress states. These eigendata comprise a complete set from which any arbitrary self-equilibrated stress state may be represented. These eigendata may be used as global functions for describing the far-field behavior in a prismatic structure. Horgan and his colleagues have solved a number of problems on homogeneous and sandwich plates under plane strain using the Airy stress function as the primary dependent variable (see, for examples, Refs. [7,8]).

For a laminated composite structure, it is more convenient to determine the eigendata numerically. Dong and Goetschel [9] developed a one-dimensional finite-element analysis for extracting eigendata for a laminated composite plate with an arbitrary number of bonded, elastic laminates. Finite-element discretization occurs in the thickness direction, see Fig. 5. Applying the theorem of minimum potential energy, a system of second-order ordinary differential equations is obtained. By invoking exponential decay $e^{-\gamma x}$, the following second-order algebraic eigenvalue problem results:

$$[K_1]\{Q\} - \gamma[K_2]\{Q\} + \gamma^2[K_3]\{Q\} = 0 \quad (4)$$

where $\{Q\}$ is an ordered set of the plate's nodal displacements. This equation is reducible to first order with a non-symmetric matrix. If a large number of degrees of freedom are involved, a Block-Stodola iteration technique [10] can be used to extract the eigendata efficiently. The solution consists of a complete set of eigenvalues and corresponding eigenvectors, which are the self-equilibrated displacement states for the given composite plate. Stresses can be computed from these displacements.

Laminated cylinders may also be solved using the same finite-element scheme, see Ref. [11]. The mechanical variables have circumferential dependence, which may be expressed analytically by Fourier series. As a circumferential mode number m occurs in this case, the counterpart to Eq. (4) for each circumferential mode has the form:

$$[K_1(m)]\{Q\} - \gamma[K_2(m)]\{Q\} + \gamma^2[K_3(m)]\{Q\} = 0 \quad (5)$$

The solution to Eq. (4) or (5) provides the global function data base for the numerical evaluation of the global stiffness matrix and the

global-local coupling as a prelude to a mathematically two-dimensional GLFEM analysis of the localized stress zones. Some preliminary results of this type have been obtained, which are contained in Refs. [12,13]. These limited scope studies indicate an overall feasibility for this approach.

The accuracy of the global functions depends on the fineness of the one-dimensional finite-element model adopted for the eigenproblem. Since only one-dimensional finite elements are used, a large model does not incur an inordinate computational effort because of a very small bandwidth.

The number of global functions required in a GLFEM analysis depends on both the nature of the localized stress and the location of the interface. Having an interface near the localized stress zone will require a larger number of global functions, but with a decrease in the finite-element coordinates. Conversely, an interface far removed from the localized zone needs fewer global functions, but is counteracted by a greater number of finite-element degrees of freedom.

Three-Dimensional Structures and the NASA Problem

A schematic of a three-dimensional prismatic structure and the NASA problem of a flat stiffened composite panel with a discontinuous stiffener are shown in Fig. 6. In this class of problems, three-dimensional finite elements must be employed in the localized stress region. Global functions must be used at the interface. They can be obtained from a two-dimensional finite-element analysis of the inverse characteristic decay lengths.

The analysis to determine the global functions follows the same methodology as that for mathematically two-dimensional structures. The prismatic cross-section is modeled by two-dimensional finite elements. With the dependence in the prismatic direction taken as e^{-YX} , an eigenproblem emerges for the extraction of eigendata that form the global function data base for the given cross section. The other aspects are the same as that described in the previous section. It is obvious that, in this case, the computational effort is greater.

Some comments can be given on a GLFEM analysis of the NASA flat panel. The set of two-dimensional global functions constitutes a complete system of eigenfunctions, with the non-zero eigenvalues associated with inverse characteristic decay lengths of self-equilibrated stress states. There are two zero eigenvalues for two stress distributions exhibiting no decay. They are the uniform axial deformation and pure bending states. These two global functions are needed in a GLFEM analysis of the NASA flat panel, since the discontinuous stiffener may produce bending in addition to its uniform end shortening. A set of global functions with all of these members present should permit a three-dimensional finite-element model to be concentrated on the details of the discontinuous stiffener region. Parametric studies wherein the hole and the gap length in the discontinuous stiffener are varied may be conducted. Each configuration will require a change of the three-dimensional finite-element mesh, but the same set of global functions may be used in all cases.

Applications to Steady-State Problems

The discussion of global functions in the two earlier sections pertained to elastostatic analysis of the localized stress zones. Here some remarks on steady-state dynamic effects are made. The one-dimensional finite-element method for generating global function data bases can be modified for steady-state inertial effects by including kinetic energy in the problem formulation. Instead of Eqs. (4) and (5), those equations become, respectively:

$$[K_1]\{Q\} - \gamma[K_2]\{Q\} + \gamma^2[K_3]\{Q\} + \omega^2[M]\{Q\} = 0 \quad (6)$$

$$[K_1(m)]\{Q\} - \gamma[K_2(m)]\{Q\} + \gamma^2[K_3(m)]\{Q\} + \omega^2[M]\{Q\} = 0 \quad (7)$$

where ω is the steady-state forcing frequency. The derivations of these equations are given in Refs. [11,14].

With these global functions, it is possible to study elastic wave scattering in prismatic structures by discontinuities during vibration or by some other steady-state dynamic input. GLFEM analysis of this type of prismatic structures will be similar to problems of elastic wave scattering by an object embedded in an infinite medium or soil-structure interaction.

Effects of Initial Stress

Using the same methodology, prismatic structures under initial stress may also be analyzed. In this case, the global functions must include the prestressing effect. One-dimensional finite-element analysis of wave propagation in laminated composite plates and cylinders under initial stress have been explored, see Refs. [15,16]. It is a straightforward task to adapt these formulations to generate an eigenproblem for the global functions for a prismatic structure under initial stress. Also, no conceptual difficulties are seen in an extension to three-dimensional prismatic structures under prestress.

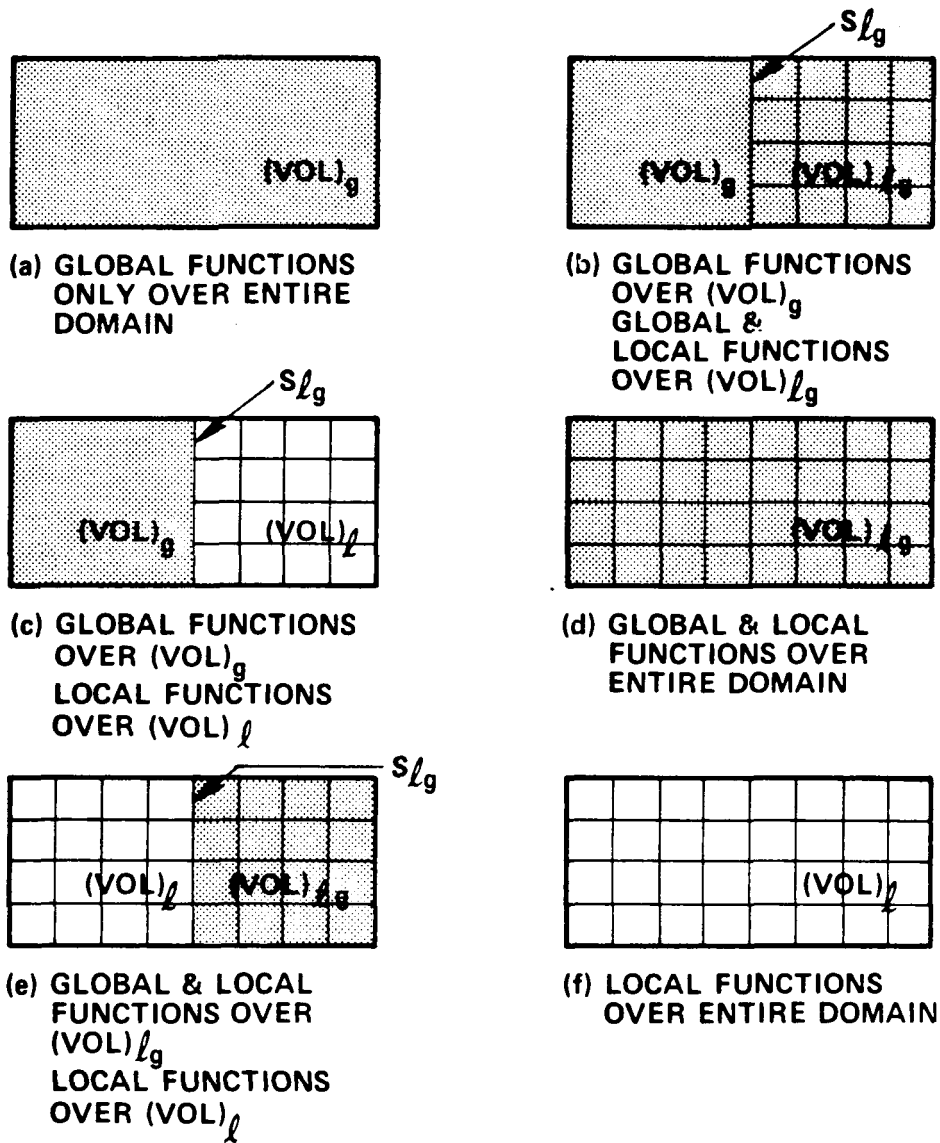
Concluding Remarks


Considerable discussion has been devoted to the strategies of GLFEM analyses of prismatic structures with localized stress regions and other discontinuities. The role of the global functions has been clearly outlined and their mathematical requirements indicated. The method for deriving these global functions for prismatic structures, whose cross-sectional geometries are complicated by laminated construction, has been discussed. From the discussion of GLFEM analysis strategy, it should be clear that GLFEM is feasible and effective. Considerable economy of computational efforts over a strictly FEM approach should be realized.


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 GLOBAL REGION $\{\underline{u}\} = [\underline{N}_g] \{\underline{s}\}$

 LOCAL (FINITE ELEMENT) REGION
 $\{\underline{u}\} = [\underline{N}_l] \{\underline{\delta}\}$


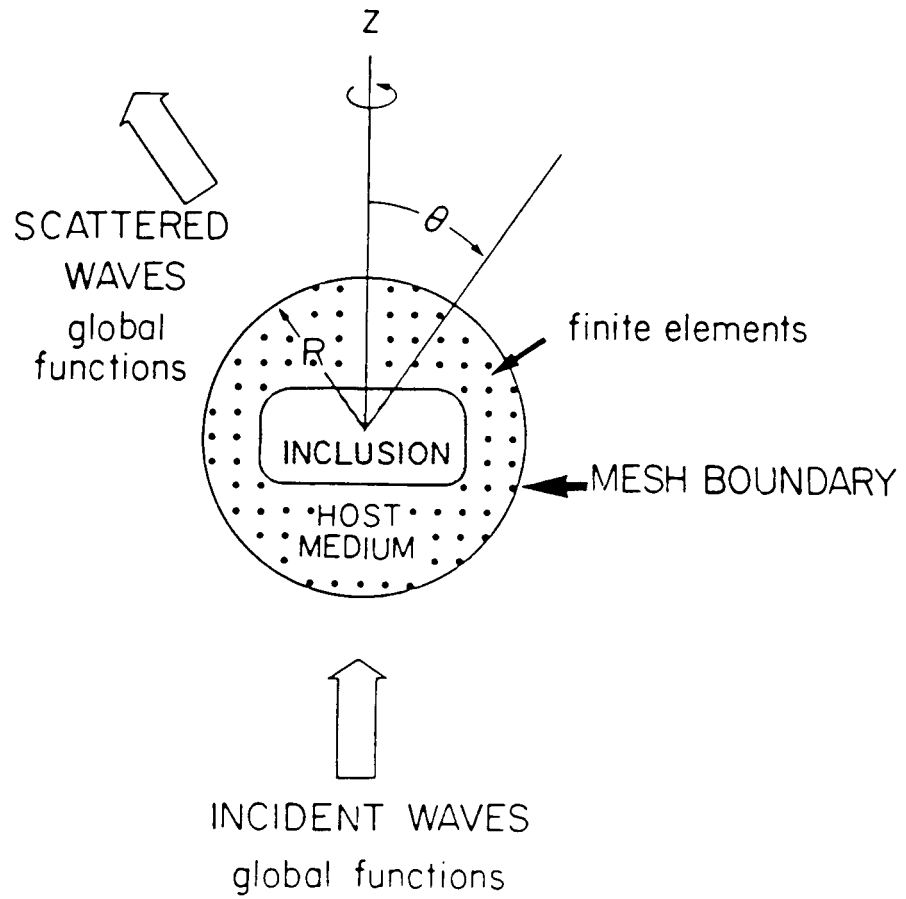
 COMBINED GLOBAL-LOCAL REGION
 $\{\underline{u}\} = [\underline{N}_l] \{\underline{\delta}\} + [\underline{N}_g] \{\underline{s}\}$

Figure 1. Basic global-local mesh configurations.



SCATTERED WAVE FIELD - SPHERICAL HARMONICS

$$u_r(\theta) = \sum_{n=0}^N \left[A_n D_{1n} - B_n S_{1n} \right] P_n(\cos\theta)$$

$$u_\theta(\theta) = \sum_{n=0}^N \left[A_n D_{2n} - B_n S_{2n} \right] \frac{dP_n(\cos\theta)}{d\theta}$$

$$\sigma_{rr}(\theta) = \sum_{n=0}^N \left[A_n D_{3n} - B_n S_{3n} \right] P_n(\cos\theta)$$

$$\sigma_{r\theta}(\theta) = \sum_{n=0}^N \left[A_n D_{4n} - B_n S_{4n} \right] \frac{dP_n(\cos\theta)}{d\theta}$$

Figure 2. Elastic wave scattering by axisymmetric inclusion embedded in an infinite isotropic medium.

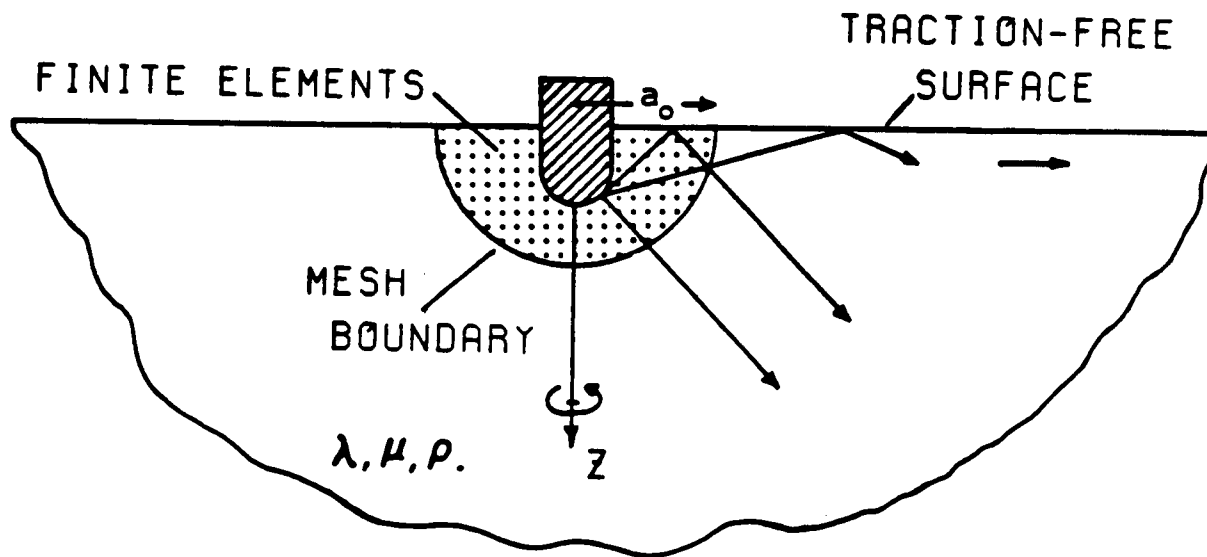


Figure 3. Soil-structure interaction problem.

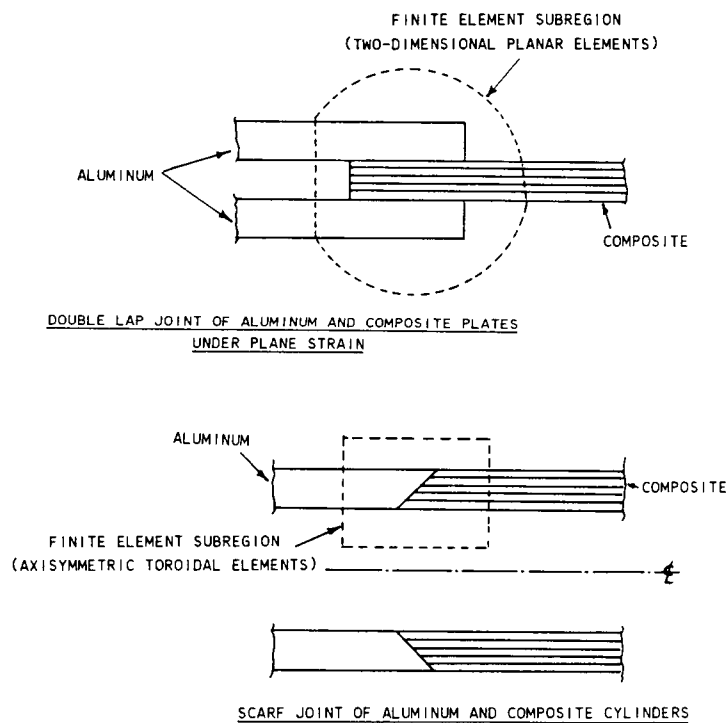


Figure 4. Two-dimensional prismatic structures.

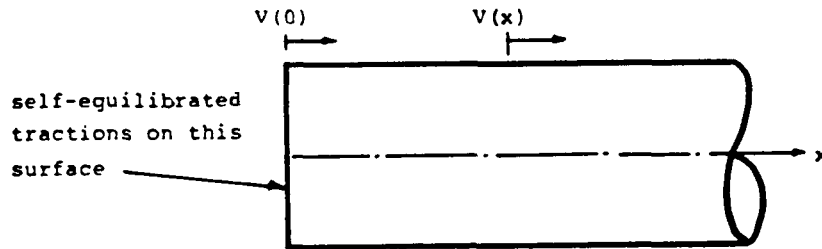
STRAIN ENERGY DECAY RATE

$$V(x) \leq V(0) \cdot \exp(-2\gamma x)$$

$$u_i(x) \leq K_1 \cdot \exp(-\gamma x)$$

$$\epsilon_{ij}(x) \leq K_2 \cdot \exp(-\gamma x)$$

$$\tau_{ij}(x) \leq K_3 \cdot \exp(-\gamma x)$$

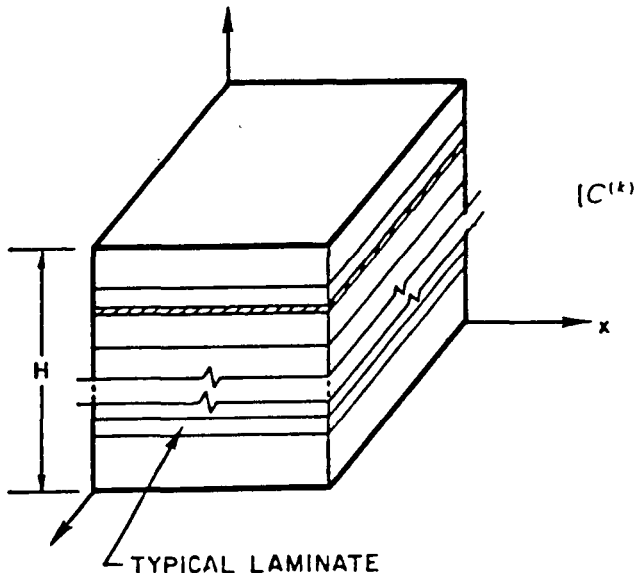


FORMULATION OF TWO-DIMENSIONAL PROBLEM FOR PLATE WITH PLANE ANISOTROPIC MATERIALS

$$\Pi = \frac{1}{2} \int \left\{ \sum_k |\epsilon|^T [C^{(k)}] |\epsilon| dy \right\} dx - \int \left\{ \tau_{xx} u(0, y) + \tau_{xy} v(0, y) \right\} dy$$

WHERE

$$\epsilon_{xx} = u_{,x}; \quad \epsilon_{yy} = v_{,y}; \quad \gamma_{xy} = u_{,y} + v_{,x}$$



DISPLACEMENT FIELD

$$u(x, y) = n_1(y) \cdot u_b(x) + n_2(y) \cdot u_m(x) + n_3(y) \cdot u_f(x)$$

$$v(x, y) = n_1(y) \cdot v_b(x) + n_2(y) \cdot v_m(x) + n_3(y) \cdot v_f(x)$$

INTERPOLATION FUNCTIONS (QUADRATIC POLYNOMIALS)

$$n_1 = 1 - 3\xi + 2\xi^2; \quad n_2 = 4\xi - 4\xi^2; \quad n_3 = 2\xi^2 - \xi$$

$$\xi = (y - y_b) / (y_f - y_b)$$

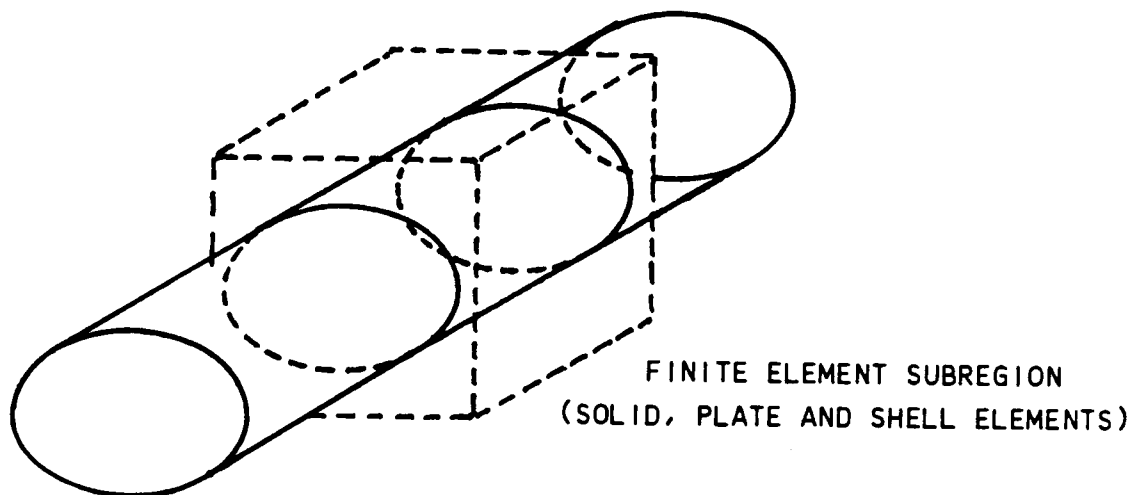
SOLUTION FORM IN X-DIRECTION

SECOND ORDER ALGEBRAIC EIGENVALUE PROB.

$$|Q| = |Q_0| \cdot \exp(-\gamma x)$$

$$[\gamma^2 [K_3] + \gamma [K_2] + [K_1]] |Q_0| = 0$$

Figure 5. Finite-element analysis of self-equilibrated edge effects in a composite plate.



EXAMPLE OF A
THREE-DIMENSIONAL PRISMATIC STRUCTURE

THE NASA FLAT PANEL WITH
DISCONTINUOUS STIFFENER

FINITE ELEMENT SUBREGION
(SOLID, PLATE AND SHELL ELEMENTS)

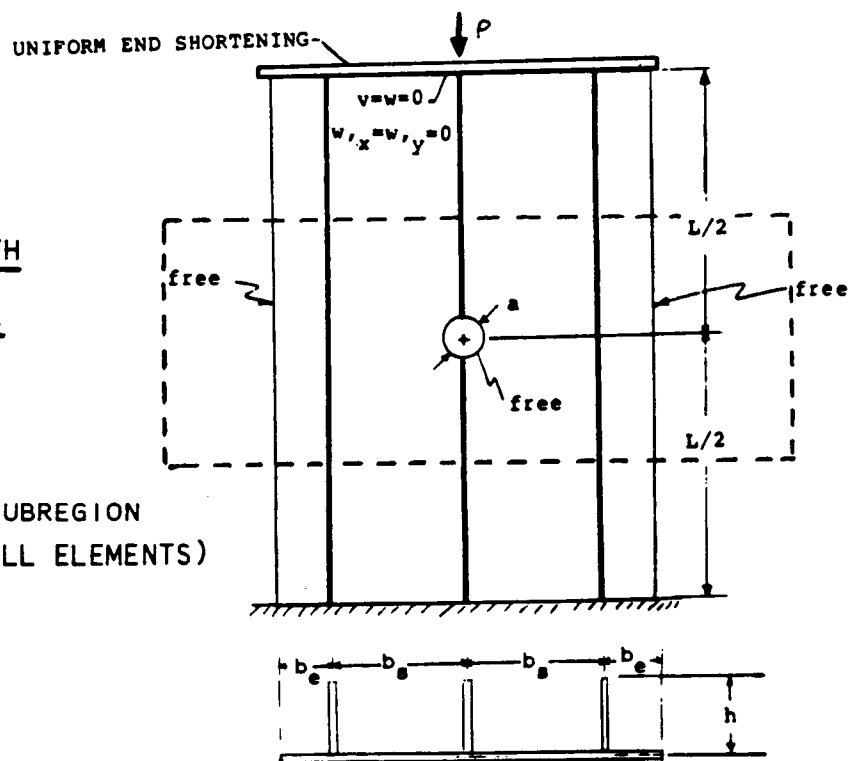


Figure 6. Three-dimensional prismatic structures.